The complexity of embedding orders into small products of chains

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THE COMPLEXITY OF EMBEDDING ORDERS INTO SMALL PRODUCTS OF CHAINS.

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Abstract. Embedding a partially ordered set into a product of chains is a classical way to encode it. Such encodings have been used in various fields such as object oriented programming or distributed computing. The embedding associates with each element a sequence of integers which is used to perform comparisons between elements. A critical measure is the space required by the encoding, and several authors have investigated ways to minimize it, which comes to embedding partially ordered sets into small products of chains. The minimum size of such an encoding has been called the *encoding dimension* [24], and the *string dimension* [22] for a slightly different definition of embeddings.

This paper investigates some new properties of the encoding dimension. We clear up the links with the string dimension and we answer the computational complexity questions raised in [22] and [24]: both these parameters are $\mathcal{NP}$-hard to compute.

1. Introduction

Partially ordered sets (*orders* for short) occur in numerous fields of computer science, like distributed computing, programming languages, databases or knowledge representation. Such applications have raised the need for storing and handling them efficiently. Many ways of encoding partially ordered sets have been proposed in the literature. Depending on the purposes, several criteria are commonly considered to guide the choice of the most appropriate encoding. One may cite the compromise between speeding up operations and saving space, the choice between dynamic or static data structures with regard to possible modifications of the order, the complexity of generating the encoding from usual data structures (like matrices or lists of successors), the restrictions on the data structures imposed by hardware and software (e.g. storing the order in a database which can be then accessed only by means of SQL requests). Performing fast comparisons between elements while saving space is the most usual issue.

Here is a non-exhaustive list of approaches that have been studied: numbering the elements in order to compress their lists of successors [1, 38], partitioning the order into nice subsets like antichains [10, 16, 44] or chains [6, 16, 29, 33, 35, 39], mixing numbering and partitioning [23, 47], seeing the order as the inclusion order on some geometrical shapes [2, 18], describing the order as the union of nice orders on the same set of elements [8, 45],

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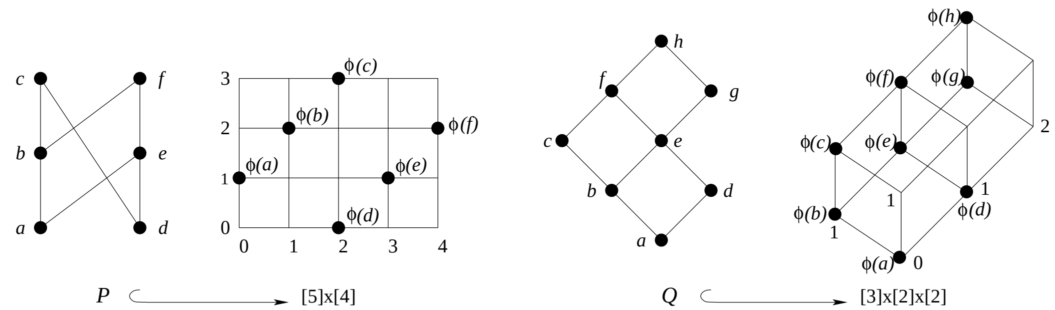


Figure 1: Examples of embeddings of the orders P and Q into some products of chains.

describing the order by combinations of boolean formulas on integer tuples [11, 19, 20, 21], focusing on lattice operations [4, 41].

Another classical scheme consists in embedding the order into another one which is known to have a nice representation. More formally, let $P = (X, \leq_P)$ be an order where $\leq_P$ is an order relation (i.e. reflexive, antisymmetric and transitive) on a ground set X, and $Q = (Y, \leq_Q)$ another order. An *order embedding* (*embedding* for short) of P into Q is a mapping φ from X into Y such that for all $x, y \in X$, $x \leq_P y$ if and only if $\varphi(x) \leq_Q \varphi(y)$. We will denote the existence of such an embedding of P into Q by $P \hookrightarrow Q$. By requiring that Q should belong to a particular class of orders, different interesting classes of embeddings can be defined.

This article focuses on finite orders and investigates a class of embeddings which have been highlighted by Habib et al [24], namely embeddings of orders into *products of chains*. Let $n \geq 1$ be an integer, a *chain* of size n is a total order with n elements. Up to an isomorphism, it can be represented by the order $\{0 < 1 < 2 < \cdots < n-1\}$ which is denoted $[n]$. Then, let $n_1, n_2, \ldots, n_d \geq 1$ be $d \geq 1$ integers, we denote by $[n_1] \times [n_2] \times \cdots \times [n_d]$ the product of the d chains where the elements are the corresponding d-uples $\{(x_1, x_2, \ldots, x_d) \mid \forall 1 \leq i \leq d,\ 0 \leq x_i \leq n_i - 1\}$ and where $(x_1, x_2, \ldots, x_d) \leq (y_1, y_2, \ldots, y_d)$ if and only if $\forall 1 \leq i \leq d$, $x_i \leq y_i$.

Any embedding φ of P into some product of chains $[n_1] \times [n_2] \times \cdots \times [n_d]$, $d \geq 1$ provides a simple way to encode P: each element $x \in X$ is labelled by its image $\varphi(x)$. Among the advantages, this information can be stored locally and no spare data structure is needed to perform the comparisons in P. Using pairwise comparisons of integers is simple enough to be implemented in various contexts. Up to small adjustments depending on the precise way the d-uples will be stored, the size of the label associated with each element is $\sum_{i=1}^{d} \lceil \log_2(n_i) \rceil$ bits. The comparison between two elements requires d comparisons of integers, that is $\mathcal{O}(d)$ time which can be shortened if they are parallelized. Figure 1 shows the embeddings of two orders into some products of chains.

As a matter of fact, such embeddings have been intensively studied when some conditions are imposed on d and the n_i's. The first important results concern the existence of those embeddings for any order. From [14, 37], it is known that any order with n elements can be embedded into some product of finite chains (of size n) and the smallest number of chains for which it works is called the *dimension* of P and denoted $dim(P)$. A lot of results about this parameter have been compiled in Trotter's book [43]. Concerning the complexity of computing $dim(P)$, Yannakakis showed in [46] that deciding whether $dim(P) \leq 3$ is $\mathcal{NP}$-complete, while deciding whether $dim(P) \leq 2$ can be done in linear time [34].

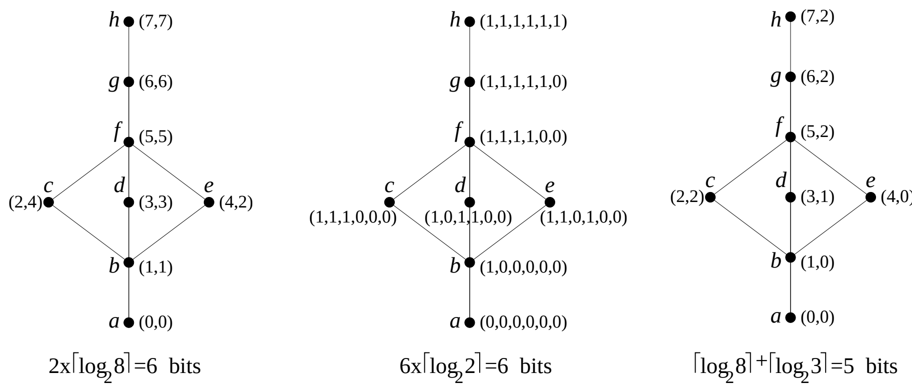


Figure 2: Three embeddings achieving respectively $dim(P)$, $dim_2(P)$ and $edim(P)$.

In [36], Novak generalized this parameter by noticing that for any $k \geq 1$ and any order P, there exists an embedding of P into some product $[k] \times [k] \times \cdots \times [k]$. It introduced the k-dimension, denoted $dim_k(P)$, as the minimum number of chains in such a product (note that $dim(P) = dim_n(P)$). The case $k = 2$, which is equivalent to label the elements by subsets of a fixed set and then consider the inclusion order, has prompted a number of theoretical studies [25, 27, 32, 42] and led to several heuristics to generate small embeddings [9, 17, 30]. However it was shown in [27] that given $k \geq 2$, deciding whether $dim_k(P) \leq d$ for arbitrary P and d is $\mathcal{NP}$-complete.

In [24, 26], Habib et al suggested to work with products of chains without any restrictions on the number d of chains and the sizes n_i of the chains. It was presented as a relaxation of the dimension and the k-dimension, and it necessarily leads to smaller labels for the elements. Indeed we always have $d \geq dim(P)$, but one may use chains much smaller than $[n]$ by using more than $dim(P)$ chains, and then save bits by comparison to the $dim(P)\lceil \log_2(n) \rceil$ bits required by the classical dimension. Consequently, they introduced a new parameter called the *encoding dimension* of P, denoted $edim(P)$, and defined as follows:

$$edim(P) = \min\{\sum_{i=1}^{d} \lceil \log_2(n_i) \rceil \mid d \geq 1,\ n_1, \ldots, n_d \geq 2,\ P \hookrightarrow [n_1] \times \cdots \times [n_d]\}$$

Figure 2 illustrates three different ways of embedding an order into some products of chains and the respective numbers of bits of their labels.

The complexity of computing $edim(P)$ was not answered in [24]. The authors conjectured that it was hard, although they believed that it might exist good approximation algorithms. In [13], de la Higuera and Nourine searched for optimal plane drawings of orders of dimension 2. When $dim(P) = 2$, one can derive from their work a polynomial algorithm to compute $\min\{\lceil \log_2(n_1) \rceil + \lceil \log_2(n_2) \rceil \mid P \hookrightarrow [n_1] \times [n_2]\}$. Note that it is not exactly $edim(P)$ since it only considers embeddings into product of *two* chains.

In their study of vector clocks in distributed systems, Garg and Skawratananand [22] introduced a class of encodings very close to order embeddings. For an order $P = (X, \leq_P$), instead of the classical definition of embeddings, they consider mappings φ from X into $[n_1] \times \cdots \times [n_d]$ such that $x <_P y$ if and only if $\varphi(x) < \varphi(y)$, where these comparisons

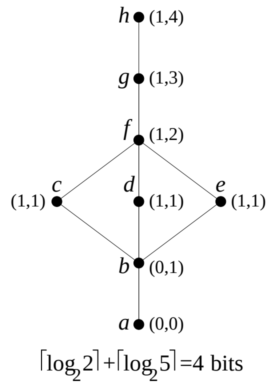


Figure 3: A string realizer of P.

are strict, but they do not require that φ is injective. Such mappings are called *string realizers* and their existence is denoted $P \rightsquigarrow [n_1] \times \cdots \times [n_d]$. Figure 3 shows a string realizer of an order (note that in this example it is not an embedding). Wishing to minimize the number of bits of the labels of elements, they introduced the *string dimension*:

$$sdim(P) = \min\{\sum_{i=1}^{d} \lceil \log_2(n_i) \rceil \mid d \geq 1,\ n_1, \ldots, n_d \geq 2,\ P \rightsquigarrow [n_1] \times \cdots \times [n_d]\}$$

(it is not actually their original definition of string dimension, but we choose to retain this one since it is their main optimization problem). Although they cite several references about the dimension of the orders and its variations, including the encoding dimension, they do not fully clear up how their string dimension relates to these former parameters.

Our paper aims at disclosing new properties of the encoding dimension. We also investigate its precise links with the string dimension. As a result, we manage to settle the computational complexity of those two parameters: both are $\mathcal{NP}$-hard to compute.

2. Definitions and notations

2.1. Partial order definitions

Let $P = (X, \leq_P)$ be an order. We only consider *finite orders* and we also denote by $|P|$ the cardinal of X. Let $x, y \in X$, $x \neq y$, we say that x and y are *comparable* in P if either $x \leq_P y$ or $y \leq_P x$. Otherwise we say that x and y are *incomparable*. An order where every pair of elements is comparable (resp. incomparable) is called a *chain* (resp. an *antichain*). By extension, for the order $P = (X, \leq_P)$ a non empty set Y of P is called a *chain* (resp. an *antichain*) of P if every pair of elements of Y is comparable (resp. incomparable) in P. The maximum cardinality of a chain of P minus 1 is called the *height* of P and denoted $h(P)$. The maximum cardinality of an antichain of P is called the *width* of P and denoted $w(P)$. An element $x \in X$ is called the *maximum* (resp. the *minimum*) of P and denoted $\max(P)$ (resp. $\min(P)$) if for all $y \in X, y \leq_P x$ (resp. $x \leq_P y$).

The *strict order relation* for $P = (X, \leq_P)$ is denoted by $<_P$ and defined for all $x, y \in X$ as $x <_P y$ if $x \leq_P y$ and $x \neq y$. For each $x \in X$, the set of *predecessors* (resp. *successors*) of x in P is defined by $Pred_P(x) = \{y \in X | y <_P x\}$ (resp. $Succ_P(x) = \{y \in X | x <_P y\}$).

Moreover we say that x is *covered* by y in P, denoted by $x \prec_P y$, if $x <_P y$ and there is no element $z \in X$ such that $x <_P z$ and $z <_P y$. To manipulate this *cover relation*, for each $x \in X$, we define the set of *immediate predecessors* (resp. *immediate successors*) of x in P which is $ImPred(x) = \{y \in X | y \prec_P x\}$ (resp. $ImSucc(x) = \{y \in X | x \prec_P y\}$).

2.2. Lattice definitions

A lattice $L = (X, \leq_L)$ is an order such that for all $x, y \in X$, the pair $\{x, y\}$ has an infimum $x \wedge_L y$ and a supremum $x \vee_L y$. For instance, the set of all the subsets of $\{1, ..., n\}$ ordered by inclusion is a lattice. It is called the *boolean lattice of dimension* n and denoted by $[2]^n$ for short since it is also isomorphic to $[2] \times [2] \times ...[2]$, n times.

From an order $P = (X, \leq_P)$, several useful lattices can be constructed such as its *Dedekind-MacNeille completion* and its *lattice of ideals*.

The *Dedekind-MacNeille completion* of P (denoted by $\mathcal{DM}(P)$)) is the unique lattice (up to an isomorphism) verifying the two properties: there exists an embedding of P into $\mathcal{DM}(P)$) and for any lattice L such that there exists an embedding of P into L, then there exists an embedding of $\mathcal{DM}(P)$) into L (see [31, 12] for proofs of its existence).

Let $P = (X, \leq)$ be an order, a set $I \subseteq X$ is an *ideal* of P if and only if $\forall x \in I$, $Pred_P(x) \subseteq I$. The set of all ideals of P ordered by inclusion is a lattice called the *ideal lattice* of P and denoted $\mathcal{I}(P)$. It is a distributive lattice (each of the operations $\wedge$ and $\vee$ is distributive with regard to the other). Conversely let $L = (X, \leq_L)$ be a distributive lattice, and $J(L) = \{j \in X \mid |ImPred(j)| = 1\}$ be the set of its *join-irreducible* elements ordered by $\leq_L$, Birkhoff's theorem [5, 12] states that L is isomorphic to $\mathcal{I}(J(L))$.

3. Basic results

3.1. Bounds on the encoding dimension

Several bounds have been established for the encoding dimension, and in a few particular cases, some exact formulas have been set. We give here an overview of the results.

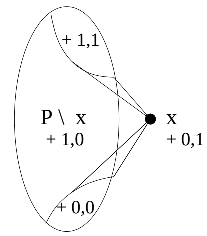
Proposition 3.1 (Folklore)**.**

1. *Let P and Q be two orders, if $P \hookrightarrow Q$ then $edim(P) \leq edim(Q)$ (Monotony).*
2. *Let P be an order and x one of its elements. Then*
$$edim(P \setminus \{x\}) \leq edim(P) \leq edim(P \setminus \{x\}) + 2 \text{ (Continuity).}$$
3. *Let P be an order, then $edim(DM(P)) = edim(P)$ (Completion).*

Proof. **Monotony**: Obviously, if $Q \hookrightarrow [n_1] \times [n_2] \times ... \times [n_d]$, then $P \hookrightarrow [n_1] \times [n_2] \times ... \times [n_d]$ and thus $edim(P) \leq edim(Q)$. **Continuity**: The left part of the inequality comes from monotony. Concerning the right part, let φ be an encoding for $P \setminus \{x\}$, by adding 2 bits we can produce an embedding φ' for P. Let φ be an optimal embedding of $P \setminus \{x\}$ into $[n_1] \times [n_2] \times ... \times [n_d]$, we define the mapping φ' from P into $[n_1] \times [n_2] \times ... \times [n_d] \times [2] \times [2]$

as follows. For all $y \in X \setminus \{x\}$, if $\varphi(y) = (\varphi_1(y), ..., \varphi_d(y))$ then

$$\varphi'(y) = \begin{cases} (\varphi(y), 0, 0) \text{ if } y <_P x, \\ (\varphi(y), 1, 1) \text{ if } y >_P x, \\ (\varphi(y), 1, 0) \text{ if } y \text{ and } x \text{ are incomparable.} \end{cases}$$



Moreover, $\varphi'(x) = (\sup_{y<_P x} \varphi_1(y), ..., \sup_{y<_P x} \varphi_d(y), 0, 1)$. One can easily check that φ' remains an embedding. The way φ is extended is illustrated above.
Completion: this is a direct consequence of the definition of $DM(P)$, since all products of chains are lattices. ■

The next propositions present some first bounds for the encoding dimension.

Proposition 3.2 (Folklore). *Let P be an order with $|P| = n$, then*

(1) $\lceil \log_2(n) \rceil \leq edim(P)$;
(2) $edim(P) \leq \lceil \log_2(k) \rceil dim_k(P)$ $\forall 2 \leq k \leq n$. *In particular, we have:*
  - *for $k = n$: $edim(P) \leq \lceil \log_2(n) \rceil dim(P) \leq \lceil \log_2(n) \rceil \frac{n}{2}$;*
  - *for $k = 2$: $edim(P) \leq dim_2(P) \leq n$.*

Proof. Since any mapping ϕ from P into $[n_1] \times [n_2] \times ... \times [n_d]$ has to be injective, we have obviously $n = |P| \leq |[n_1] \times [n_2] \times ... \times [n_d]| = n_1 \times n_2 \times ... \times n_d$. Thus $\log_2(n) \leq \sum_{i=1}^{d} \log_2(n_i) \leq \sum_{i=1}^{d} \lceil \log_2(n_i) \rceil$. The upper bound involving dim_k is directly derived from the definition of dim_k, and the bound $dim(P) \leq n/2$ is Hiragushi's inequality [28, 43]. ■

Proposition 3.3 ([24]). *Let P be an order, $\mathcal{C}(P)$ be the set of partitions into chains of P then*

$$edim(P) \leq \min\{\sum_{i=1}^{\ell} \lceil \log_2 (|C_i| + 1) \rceil \mid \{C_1, ..., C_\ell\} \in \mathcal{C}(P)\}$$

Consequently, $edim(P) \leq w(P) \lceil \log_2(h(P) + 2) \rceil$.

3.2. Classes of orders

For the class of distributive lattices, i.e. ideal lattices, an exact formula is given in [24]. It is related to Proposition 3.3.

Proposition 3.4 ([6, 7, 24]). *Let P be an order, $\mathcal{C}(P)$ be the set of partitions into chains of P then*

$$edim(\mathcal{I}(P)) = \min\{\sum_{i=1}^{\ell} \lceil \log_2 (|C_i| + 1) \rceil \mid \{C_1, ..., C_\ell\} \in \mathcal{C}(P)\}$$

Proposition 3.5.

(1) *Let A_n be the antichain of size n, then $edim(A_n) = sp(n) = \min\{d \mid \binom{d}{\lfloor d/2 \rfloor} \geq n\}$;*
(2) *Let $[n]$ be the chain of size n, then $edim(P) = \lceil \log_2 (n) \rceil$;*
(3) *Let $[2]^n$ be the boolean lattice of dimension n, then $edim([2]^n) = n$;*
(4) *Let $[n_1] \times ... \times [n_d]$ be a product of chains, then $edim([n_1] \times ... \times [n_d]) = \sum_{i=1}^{d} \lceil \log_2 (n_i) \rceil$.*

Proof.

(1) $edim(A_n) = sp(n)$: We can show that $edim(A_n) = dim_2(A_n)$. Let φ be an embedding of A_n into $[n_1] \times [n_2] \times ... \times [n_d]$ with $\varphi(x) = (\varphi_1(x), \varphi_2(x), ..., \varphi_d(x))$. Suppose that $n_1 \geq 3$. Then we can produce an embedding $\widetilde{\varphi}$ of A_n into $[2] \times [\lceil \frac{n_1}{2} \rceil] \times [n_2] \times ... \times [n_d]$ defined by $\widetilde{\varphi}(x) = (\widetilde{\varphi_0}(x), \widetilde{\varphi_1}(x), ..., \widetilde{\varphi_d}(x))$, where if $0 \leq \varphi_1(x) < \frac{n_1}{2}$, then $\widetilde{\varphi_0}(x) = 0$ and $\widetilde{\varphi_1}(x) = \varphi_1(x)$. And otherwise if $\frac{n_1}{2} \leq \varphi_1(x) \leq n_1$, then $\widetilde{\varphi_0}(x) = 1$ and $\widetilde{\varphi_1}(x) = \varphi_1(x) - \lceil \frac{n_1}{2} \rceil$. It can be easily checked that $\widetilde{\varphi}$ remains an embedding. Moreover this transformation does not increase the size of the embedding. We only have to check that $\lceil \log_2(n_1) \rceil$ is larger or equal to $1 + \lceil \log_2(\lceil \frac{n_1}{2} \rceil) \rceil = \lceil \log_2(2 \lceil \frac{n_1}{2} \rceil) \rceil$. These values are actually equal since $\forall k \in \mathbb{N}$, $2 \lceil \frac{n_1}{2} \rceil \leq 2^k \iff \lceil \frac{n_1}{2} \rceil \leq 2^{k-1} \iff \frac{n_1}{2} \leq 2^{k-1} \iff n_1 \leq 2^k$.

By starting from φ an optimal embedding of A_n and repeatedly applying this transformation , we obtain an optimal embedding of A_n into a product where all the chains are $[2]$. The smallest size of an embedding of A_n into some product $[2]^d$ is by definition $d = dim_2(A_n)$ and thanks to Sperner's Theorem [3, 15] it is known that $dim_2(n) = sp(n)$.

(2) Let $[n]$ be the chain of size n, then $\lceil \log_2 (n) \rceil \leq edim([n])$ from Proposition 3.2. We also have $edim([n]) \leq \lceil \log_2 (n) \rceil dim(P)$ from Proposition 3.2. Since $[n]$ is a chain, we have $dim(P) = 1$. Finally $edim([n]) = \lceil \log_2 (n) \rceil$;

(3) Let $[2]^n$ be a boolean lattice, we have $n = \lceil \log_2 ([2]^n) \rceil \leq edim([2]^n)$ from Proposition 3.2. We also have $edim([2]^n) \leq dim_2([2]^n) = n$. Finally $edim([2]^n) = n$;

(4) The formula for $P = [n_1] \times [n_2] \times ... \times [n_d]$ is not as obvious as it may look. Proposition 3.2 provides the lower bound $\lceil \sum_{i=1}^{d} (n_i) \log_2(n_i) \rceil \leq edim(P)$ and since $P \hookrightarrow [n_1] \times [n_2] \times ... \times [n_d]$, by definition of $edim(P)$, we have the upper bound $edim(P) \leq \sum_{i=1}^{d} \lceil \log_2(n_i) \rceil$. Unfortunately these two bounds are not equal in general. The exact formula requires more arguments.

The order P is clearly a distributive lattice and the order induced on its join irreducible elements is $[n_1 - 1] \cup [n_2 - 1] \cup ... \cup [n_d - 1]$. From Proposition 3.4, we have $edim(\mathcal{I}(P)) = \min\{\sum_{i=1}^{d} \lceil \log_2 (|C_i| + 1) \rceil \mid \{C_1, ..., C_d\} \in \mathcal{C}(\mathcal{I}(P))\}$. If we choose $C_i = [n_i - 1]$ for all $1 \leq i \leq d$, we obtain an embedding of size $\sum_{i=1}^{d} \lceil \log_2(n_i - 1 + 1) \rceil = \sum_{i=1}^{d} \lceil \log_2 n_i \rceil$. Now any other partition into chains of $[n_1 - 1] \cup [n_2 - 1] \cup ... \cup [n_d - 1]$ has the form $\{C_{i,j}\}_{1 \leq i \leq d,\ 1 \leq j \leq J_i}$ where $\{C_{i,1}, C_{i,2}, ..., C_{i,J_i}\}$ is a partition of $[n_i - 1]$. To bound the size of the associated embedding, we use the following lemma.

Lemma 3.6. *The function $x \mapsto \lceil \log_2 (x + 1) \rceil$ is non-decreasing and sub-additive over $\mathbb{R}_+$.*

Proof. The functions $x \mapsto \lceil x \rceil$ is non-decreasing and sub-additive, as well as $x \mapsto \log_2 (x+1)$ where sub-additivity comes from concavity. So their composition is also non-decreasing and sub-additive. It means that for any finite sequence of positive reals (x_i), we have $\lceil \log_2 (\sum_i x_i + 1) \rceil \leq \sum_i \lceil \log_2 (x_i + 1) \rceil$. ∎

Due to this sub-additivity we have for all $1 \leq i \leq d$, $\lceil \log_2(\sum_j |C_{i,j}| + 1) \rceil \leq \sum_j \lceil \log_2(|C_{i,j}| + 1) \rceil$, that is $\lceil \log_2 n_i \rceil \leq \sum_j \lceil \log_2(|C_{i,j}| + 1) \rceil$. Thus $\sum_i \lceil \log_2 n_i \rceil \leq \sum_{i,j} \lceil \log_2(|C_{i,j}| + 1) \rceil$ This shows that the minimum size of an embedding of $[n_1 - 1] \times [n_2 - 1] \times ... \times [n_d - 1]$ is $\sum_i \lceil \log_2 n_i \rceil$.

∎

3.3. Links with the string dimension

It turns out that the string dimension and the encoding dimension are exactly the same parameter up to a small decomposition step. Let $P = (X, \leq_P)$ be an order, $x, y \in X$ are *twins* if $Succ_P(x) = Succ_P(y)$ and $Pred_P(x) = Pred_P(y)$ (they are necessarily incomparable). They are also called *duplicated holdings* in [43]. Being twins is an equivalence relation denoted $\sim$ which can be used to quotient the order P (i.e. identify each set of twins by a single element). The quotiented order on the quotiented set $X/\sim$ is denoted $P/\sim$.

Proposition 3.7. *Let $P = (X, \leq_P)$ be an order. Then $sdim(P) = edim(P/\sim)$. Conversely, if P has a minimum and a maximum, then $edim(P) = sdim(P \times [2]) - 1$.*

Proof. To prove the first equality, let φ be an embedding of $P/\sim$ into $[n_1] \times \cdots \times [n_d]$. Let $\widetilde{x} \in X/\sim$, label all the elements belonging to $\widetilde{x}$ by $\varphi(\widetilde{x})$. It clearly provides a string realizer of P. Inversely, let φ be a string realizer mapping P into $[n_1] \times \cdots \times [n_d]$. Let $x, y \in X$ such that $\varphi(x) = \varphi(y)$. From the definition of a string realizer, x and y are necessarily twins. Then let $\widetilde{x} \in X/\sim$, label $\widetilde{x}$ by $\varphi(y)$ for an arbitrary $y \in \widetilde{x}$. This mapping of $P/\sim$ into $[n_1] \times \cdots \times [n_d]$ is clearly an injective string realizer, that is an order embedding.

For the second inequality, note that $P \times [2]$ has no twins, thus from the first equality, we have $sdim(P \times [2]) = edim(P \times [2])$. Since $P \hookrightarrow P \times [2]$, we have $edim(P) \leq edim(P \times [2])$ by monotony. Moreover, from any embedding of P into $[n_1] \times \cdots \times [n_d]$, one can easily construct an embedding of $P \times [2]$ into $[n_1] \times \cdots \times [n_d] \times [2]$. It adds only one bit, thus $edim(P \times [2]) \leq edim(P) + 1$. For now, we only have $edim(P) \leq edim(P \times [2]) \leq edim(P) + 1$.

Suppose that P has a minimum m and a maximum M, $m \neq M$. Consider an embedding $\varphi = (\varphi_1, \ldots, \varphi_d)$ of $P \times [2]$ into $[n_1] \times \cdots \times [n_d]$. The elements $(m, 1)$ and $(M, 0)$ of $P \times [2]$ are incomparable. Since $(m, 1) \not\leq (M, 0)$, there exists $1 \leq i \leq d$ such that $\varphi_i((m, 1)) > \varphi_i((M, 0))$. Since φ is an embedding, for all $x, y \in X$, we have $\varphi_i((x, 1)) \geq \varphi_i((m, 1)) > \varphi_i((M, 0)) \geq \varphi_i((y, 0))$. To construct a smaller embedding for P, choose from $\varphi_i((X, 0))$ or $\varphi_i((X, 1))$ the set using the smallest range of integers. If $\varphi_i((M, 0)) < \lfloor n_i/2 \rfloor$, then consider the mapping $\varphi' = (\varphi'_1, \ldots, \varphi'_d)$ from X into $[n_1] \times \cdots \times [\lfloor \frac{n_i}{2} \rfloor] \times \cdots \times [n_d]$ defined by: $\forall 1 \leq j \leq d$, $\varphi'_j(x) = \varphi_j((x, 0))$. It is an order embedding. Otherwise consider the mapping $\varphi' = (\varphi'_1, \ldots, \varphi'_d)$ from X into $[n_1] \times \cdots \times [\lfloor \frac{n_i}{2} \rfloor] \times \cdots \times [n_d]$ defined by: $\forall 1 \leq j \leq d$, $j \neq i$, $\varphi'_j(x) = \varphi_j((x, 1))$ and $\varphi'_i(x) = \varphi_i(x) - \lfloor \frac{n_i}{2} \rfloor$. In both case, φ' use one bit less than φ. Consequently $edim(P) \leq edim(P \times [2]) - 1$ which completes the proof. ■

Remark 3.8. We conjecture that $edim(P) = sdim(P \times [2]) - 1$ holds for any order P.

Remark 3.9. As expected, the encoding dimension and the string dimension are very close measures. One can easily derive bounds for one measure from bounds for the other measure. Note that in [22], the authors provide the upper bound $sdim(P) \leq w(P) \lceil \log_2(h(P) + 2) \rceil$ which corresponds to the same bound presented in Section 3 for $edim(P)$. However they claim that a corollary of their proof is: $\forall k \geq h(P) + 2$, $dim(P) = dim_k(P)$. This statement is clearly false: consider A_n the antichain of size n, $h(A_n) = 0$, $dim(A_n) = 2$ but, as recalled in Section 3.2, $dim_2(A_n) = sp(n) = \min\{d \mid \binom{d}{\lfloor d/2 \rfloor} \geq n\}$ which tends to $+\infty$ when n tends to $+\infty$.

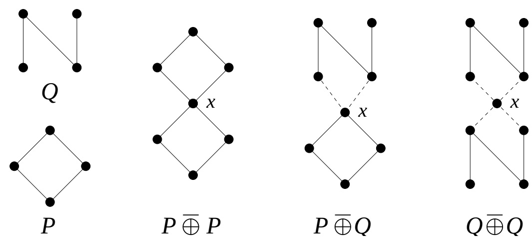


Figure 4: Examples of application of $\overline{\oplus}$ (the *join element* is denoted by x).

4. NP-Completeness

4.1. Definitions and operations

We first give back the classical *series* operation on orders, denoted $\oplus$.

Definition 4.1 ($\oplus$ operator)**.** Let $P = (X, \leq_P)$ and $Q = (Y, \leq_Q)$ be two orders s.t. $X \cap Y = \emptyset$, the $\oplus$ operator is defined as follows: $P \oplus Q = (X \cup Y, \leq_{P \oplus Q})$ where for all $u, v \in X \cup Y$,

$$u \leq_{P\oplus Q} v \text{ if } \begin{cases} (u, v \in X, \text{ and } u \leq_P v), \text{ or} \\ (u, v \in Y, \text{ and } u \leq_Q v), \text{ or} \\ (u \in X \text{ and } v \in Y). \end{cases}$$

We now introduce a slight variation of this operator.

Definition 4.2 ($\overline{\oplus}$ operator)**.** Let $P = (X, \leq_P)$ and $Q = (Y, \leq_Q)$ be two orders s.t. $X \cap Y = \emptyset$, the $\overline{\oplus}$ operator is defined as follows (*max* stands for maximum and *min* for minimum):

$$P \,\overline{\oplus}\, Q = \begin{cases} P \oplus [1] \oplus Q \text{ if } P \text{ without max and } Q \text{ without min,} \\ P \oplus Q \text{ if } (P \text{ without max and } Q \text{ with min) or } (P \text{ with max and } Q \text{ without min),} \\ P \setminus \{max(P)\} \oplus Q \text{ if } P \text{ with max and } Q \text{ with min,} \end{cases}$$

Figure 4 illustrates the result of $\overline{\oplus}$ in the three cases. In $P\overline{\oplus}Q$, the element between P and Q is called the *join element*.

Definition 4.3 (Spaces of embeddings)**.** Let P be an order, we define the spaces of chain products in which P can be embedded as follow:

$$E(P) = \{(n_1, n_2, ..., n_d) \in \mathbb{N}^d \mid d \geq 1,\ P \hookrightarrow [n_1] \times [n_2] \times ... \times [n_d]\}$$

Another set is associated with P, where the finite sequences in $E(P)$ are completed with an infinite sequence of 1:

$$\widetilde{E}(P) = \{(n_1, n_2, ..., n_d, 1, 1, ...) \in \mathbb{N}^{\mathbb{N}^*} \mid (n_1, n_2, ..., n_d) \in E(P)\}$$

By ordering the set of n_i decreasingly we get two new definitions:
$E_{\geq}(P) = \{(n_1, n_2, ..., n_d) \in \mathbb{N}^d \mid (n_1, n_2, ..., n_d) \in E(P) \text{ and } n_1 \geq n_2 ... \geq n_d\}$
$\widetilde{E}_{\geq}(P) = \{(n_1, n_2, ..., n_d, 1, 1, ...) \in \mathbb{N}^{\mathbb{N}^*} \mid (n_1, n_2, ..., n_d) \in E_{\geq}(P)\}$

Definition 4.4 (Volume of P)**.** Let P be an order then

$$vol(P) = \min\{\prod_{i=1}^{d} n_i \mid (n_i)_{1 \leq i \leq d} \in E(P)\}$$

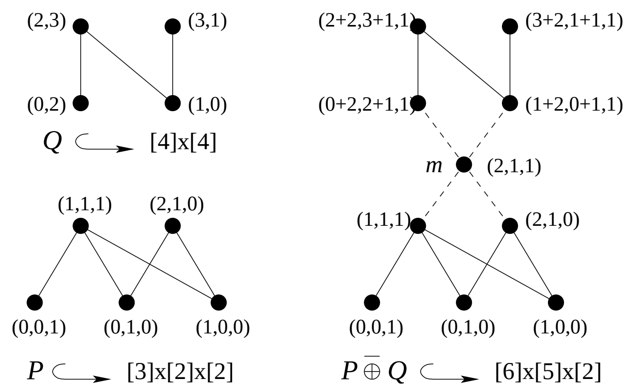


Figure 5: Constructing an embedding of $P\overline{\oplus}Q$ from embeddings of P and Q (m is the join element).

Remark 4.5. The volume of an order is closely related to its encoding dimension up to rounding effects: we clearly have $\log_2(vol(P)) = \min\{\sum_{i=1}^{d} \log_2(n_i) \mid (n_1, n_2, ..., n_d) \in E(P)\} \leq edim(P)$. The equality holds if, for some optimal embedding, all the n_i are powers of 2.

We have introduced the infinite sequences $\widetilde{E}(P)$ together with $E(P)$ for commodity. It will simplify the description of the embeddings of $P\overline{\oplus}Q$ which consist in pairing embeddings of P and Q even when they use different numbers of chains. Moreover it preserves the definition of volumes since we clearly have $vol(P) = \min\{\prod_{i=1}^{+\infty} n_i \mid (n_i)_{i\geq 1} \in \widetilde{E}(P)\}$.

Proposition 4.6. *Let $P = (X, \leq_P)$ and $Q = (Y, \leq_Q)$ be two orders, then:*

$$\widetilde{E}(P\overline{\oplus}Q) = \{(n_i + n'_i - 1)_{i\geq 1} \mid (n_i)_{i\geq 1} \in \widetilde{E}(P) \text{ and } (n'_i)_{i\geq 1} \in \widetilde{E}(Q)\}$$

Consequently,

$$vol(P\overline{\oplus}Q) = \min\{\prod_{i=1}^{+\infty}(n_i + n'_i - 1) \mid (n_i)_{i\geq 1} \in \widetilde{E}(P) \text{ and } (n'_i)_{i\geq 1} \in \widetilde{E}(Q)\}$$

Proof. First, let $(n_i)_{i\geq 1} \in \widetilde{E}(P)$ and $(n'_i)_{i\geq 1} \in \widetilde{E}(Q)$. Let us show that $(n_i + n'_i + 1)_{i\geq 1}$ belongs to $\widetilde{E}(P\overline{\oplus}Q)$ There exist d and d' such that $P \hookrightarrow [n_1] \times \cdots \times [n_d]$ and $Q \hookrightarrow [n'_1] \times \cdots \times [n'_{d'}]$. We denote $D = \max(d, d')$. Let φ (resp. φ') be an embedding of P (resp. Q) into $[n_1] \times \cdots \times [n_D]$ (resp. $[n'_1] \times \cdots \times [n'_D]$). Then we can construct the following mapping from $P\overline{\oplus}Q$ into $[n_1 + n'_1 - 1] \times \cdots \times [n_D + n'_D - 1]$ with:

$$\Phi(x) \;=\begin{cases}\varphi(x) \text{ if } x \in X \setminus \max(P) \text{ (if it exists)},\\ (\varphi'_1(x) + n_1 - 1, ..., \varphi'_D(x) + n_D - 1) \text{ if } x \in Y \setminus \min(Q) \text{ (if it exists)},\\ (n_1 - 1, ..., n_D - 1) \text{ for the join element},\end{cases} \tag{4.1}$$

One can easily check that $\Phi(x)$ is an order embedding (see Fig. 5 for an example).

Conversely, let $(N_i)_{i\geq 1} \in \widetilde{E}(P\overline{\oplus}Q)$. There exists $D \geq 1$ and Φ an embedding of $P\overline{\oplus}Q$ into $[N_1] \times \cdots \times [N_D]$. Let m be the join element of $P\overline{\oplus}Q$, $\Phi(m) = (m_1, ..., m_D)$. On one hand, Φ restricted to X is an embedding of P into $[m_1 + 1] \times \cdots \times [m_D + 1]$. On the other hand, for $x \in Y$ we can set $\phi'(x) = (\Phi_1(x) - m_1, ..., \Phi_D(x) - m_D)$. Each component i of $\phi'(x)$ belongs to the interval $[0, N_i - 1 - m_i]$. Once again, one can easily check that ϕ' is an embedding of Q into $[N_1 - m_1] \times \cdots \times [N_D - m_D]$. In this way, $(N_i)_{i\geq 1}$ has the form $(n_i + n'_i - 1)_{i\geq 1}$ with $(n_i)_{i\geq 1} \in \widetilde{E}(P)$ and $(n'_i)_{i\geq 1} \in \widetilde{E}(Q)$ by setting $n_i = m_i + 1$ and $n'_i = N_i - m_i$. ∎

Remark 4.7. Note that we focus on $\widetilde{E}(P\overline{\oplus}Q)$, but one can check that the same reasoning apply to $\widetilde{E}(P \oplus Q)$. As a result, $\widetilde{E}(P \oplus Q) = \widetilde{E}(P\overline{\oplus}Q)$ unless P has a maximum and Q has a minimum. In this latter case, $\widetilde{E}(P \oplus Q) = \{(n''_i)_{i\geq 1} \mid (n_i)_{i\geq 1} \in \widetilde{E}(P),\ (n'_i)_{i\geq 1} \in \widetilde{E}(Q)$ and $\exists j \in \mathbb{N}^*,\ \forall i \neq j,\ n''_i = n_i + n'_i - 1,$ and $n''_j = n_j + n'_j\}$.

In fact one can refine the latest formula on $vol(P\overline{\oplus}Q)$: as shown below the optimum is necessarily reached for embeddings which are paired by matching larger chains together.

Proposition 4.8. *Let P and Q be two partial orders, then*

$$vol(P\overline{\oplus}Q) = \min\{\prod_{i=1}^{+\infty}(n_i + n'_i - 1) \mid (n_i)_{i\geq 1} \in \widetilde{E}_{\geq}(P) \text{ and } (n'_i)_{i\geq 1} \in \widetilde{E}_{\geq}(Q)\}$$

.

Proof. Let $(n_i)_{i\geq 1} \in \widetilde{E}(P)$ and $(n'_i)_{i\geq 1} \in \widetilde{E}(Q)$, the formula in Proposition 4.6 calls for the minimum of $\prod_{i=1}^{d}(m_i + m'_i - 1)$ for all the permutations $(m_i)_{i\geq 1}$ (resp. $(m'_i)_{i\geq 1}$) of $(n_i)_{i\geq 1}$ (resp. $(n'_i)_{i\geq 1}$). However one can show that this minimum is reached by sorting $(n_i)_{i\geq 1}$ and $(n'_i)_{i\geq 1}$ decreasingly. This is proved thanks to the following lemma.

Lemma 4.9. *Let $a, b, a', b' \in \mathbb{R}$ s.t. $a \geq b$ and $a' \geq b'$. Then*

$$(a + a' - 1)(b + b' - 1) \leq (a + b' - 1)(a' + b - 1)$$

This lemma can be easily checked by expanding the expressions. It ensures that if $(m_i)_{i\geq 1}$ and $(m'_i)_{i\geq 1}$ are two permutations achieving the minimum $\prod_{i=1}^{+\infty}(m_i + m'_i - 1)$, and if m_I and $m'_{I'}$ are the respective maximum of $(m_i)_{i\geq 1}$ and $(m'_i)_{i\geq 1}$, by transforming the associations $m_I + m'_I - 1$ and $m_{I'} + m'_{I'} - 1$ into $m_I + m'_{I'} - 1$ and $m_{I'} + m'_I - 1$, the product does not increase. By repeating this kind of exchanges which pair the values of $(n_i)_{i\geq 1}$ and $(n'_i)_{i\geq 1}$ decreasingly, one preserves a minimum product. Figure 6 illustrates this property.

Consequently the formula for the volume only requires the pairings of the decreasing sequences in $\widetilde{E}(P)$ and $\widetilde{E}(Q)$, that is $\widetilde{E}_{\geq}(P)$ and $\widetilde{E}_{\geq}(Q)$. ∎

4.2. Reduction

The next theorem and its corollary set the complexities of computing the encoding dimension and the string dimension.

Theorem 4.10. *The decision problem associated to the* encoding dimension*, i.e. deciding whether $edim(P) \leq k$ for arbitrary P and k, is $\mathcal{NP}$-complete.*

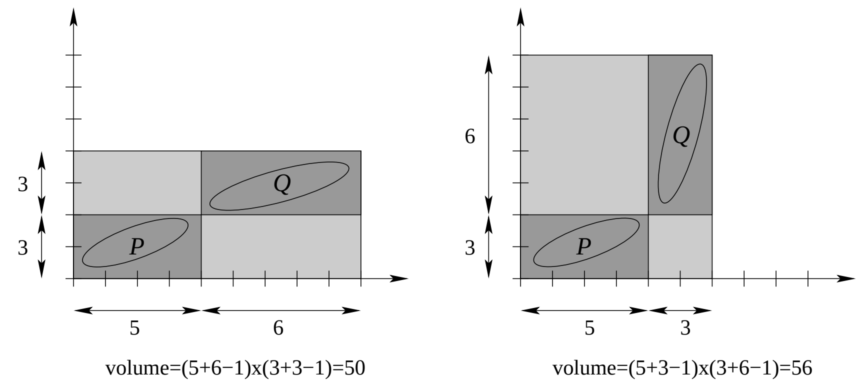


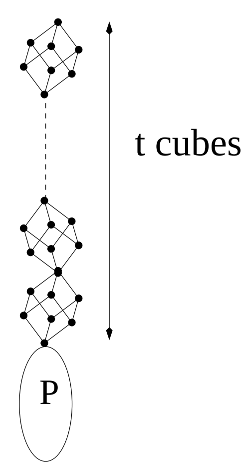
Figure 6: Comparing associations of embeddings of P and Q to get the smaller embedding of $P\overline{\oplus}Q$.

Proof. This problem belongs to $\mathcal{NP}$. Let $P = (X, \leq_P)$ an order with n elements and k an integer, note first that if $k > n$, we necessarily have $edim(P) \leq k$ by Proposition 3.2. Now suppose that $k \leq n$, a certificate for P and k is composed of an integer $d \geq 1$, a d-uple $(n_1, \ldots, n_d)$ of integers such that $\forall\ 1 \leq i \leq d$, $n_i \geq 2$ and $\sum_{i=1}^{d} \lceil \log_2(n_i) \rceil \leq k$, and a mapping φ from X to $[n_1] \times \cdots \times [n_d]$. Since $d \leq n$ due to constraints and $k \leq n$ by assumption, the size of this certificate is polynomial with the size of P. Then checking whether φ is an order embedding can be done in $\mathcal{O}(dn^2)$ time.

Now we present a reduction from the decision problem associated to the *dimension of orders*, i.e. given an order P deciding whether $dim(P) \leq 3$, which was proved $\mathcal{NP}$-complete by Yannakakis [46]. Let the order P be an instance for the dimension, we first construct an order Q equal to

$$P \ \overline{\oplus}\ \underbrace{[2]^3 \overline{\oplus} [2]^3 ... \overline{\oplus} [2]^3}_{t\ times}$$

as illustrated on the right.



Claim 1. For any order P, if $dim(P) \leq 3$ then $edim(Q) \leq 3\lceil \log_2(n+t) \rceil$.
Indeed, if $dim(P) \leq 3$ then $(n, n, n) \in E(P)$. By applying t times Proposition 4.6 with $(2, 2, 2) \in E([2]^3)$, we obtain that $(n + t \cdot (2-1), n + t \cdot (2-1), n + t \cdot (2-1)) \in E(Q)$. In other words, $Q \hookrightarrow [n+t] \times [n+t] \times [n+t]$. Thus $edim(Q) \leq 3\lceil \log_2\ (n+t) \rceil$.
Claim 2. For any order P, if $dim(P) \geq 4$ then $edim(Q) \geq 1 + 3\log_2(t+2)$;
Indeed, if $dim(P) \geq 4$ and $(n_1, n_2, ..., n_d) \in E_{\geq}(P)$ then, for $1 \leq i \leq 4$, we have $n_i \geq 2$. By applying t times Proposition 4.8 with $(2, 2, 2) \in [2]^3$, we obtain that $vol(Q) \geq 2(2+t)^3$. Since $edim(Q) \geq \log_2\ vol(Q)$, we have $edim(Q) \geq 1 + 3\log_2\ (t+2)$.

To sum up, we have:

(1) If $dim(P) \leq 3$ then $edim(Q) \leq 3\lceil \log_2(n+t) \rceil = f(t)$;
(2) If $dim(P) \geq 4$ then $edim(Q) \geq 1 + 3\log_2(t+2) = g(t)$;

Asymptotically $f(t)$ is strictly smaller than $g(t)$ which is a good news, since $f(t) < g(t)$ ensures that $edim(Q) \leq 3$ if and only if $dim(P) \leq 3$. Thus given P we now look for t large enough to have $f(t) < g(t)$ but small enough so that the instance $(Q, f(t))$ for the encoding dimension keeps a reasonable size.

We study $g(t) - f(t) = 1 + 3\log_2(t+2) - 3\lceil \log_2(n+t) \rceil$. We choose to restrict ourselves to t s.t. $t + n = 2^s$, $s \in \mathbb{N}$. Then $g(t) - f(t) = 1 + 3\log_2(2^s - n + 2) - 3s = 1 + 3\log_2(1 - \frac{n-2}{2^s})$. Since for all $0 \leq \epsilon \leq \frac{1}{2}$, $\log_2(1-\epsilon) \geq -2\epsilon$, if $\frac{n-2}{2^s} \leq \frac{1}{2}$, then $g(t) - f(t) \geq 1 - 6\frac{n-2}{2^s}$. Thus a sufficient condition for $g(t) - f(t) > 0$ is $2^s > 6(n-2)$. To sum up, we wish that $t + n$ is a power of 2 and $t + n > 6n - 12$, i.e. $t + n \geq 6n - 11$. We can choose $t + n = 2^{\lceil \log_2(6n-11) \rceil}$. Then $t = 2^{\lceil \log_2(6n-11) \rceil} - n \leq 2 \times 2^{\log_2(6n-11)} - n = 11n - 22$ which implies that $|Q| \leq |P| + 8t \leq n + 8(11n - 22)$. Finally by choosing this value for t, we can decide whether $dim(P) \leq 3$ by checking if $edim(Q) \leq f(t)$ and our reduction is polynomial with the size of P. ∎

Corollary 4.11. *The decision problem associated to the* string dimension*, i.e. deciding whether $sdim(P) \leq k$ for arbitrary P and k, is $\mathcal{NP}$-complete.*

Proof. The problem is in $\mathcal{NP}$ with arguments similar to the ones for the encoding dimension. Now we use a reduction from the encoding dimension. Let P be an order and $k \geq 1$ an instance for the encoding dimension. Add a maximum, as well as a minimum, to P if it is missing. This order $\widehat{P}$ (which is $[1]\overline{\oplus}P\overline{\oplus}[1]$) clearly satisfies $edim(\widehat{P}) = edim(P)$. From Proposition 3.7, $sdim(\widehat{P} \times [2]) = edim(P) + 1$. Thus $edim(P) \leq k$ if and only if $sdim(\widehat{P} \times [2]) \leq k + 1$, and the instance $\widehat{P} \times [2]$ and $k + 1$ for the string dimension is polynomial with the size of the initial instance. ∎

Remark 4.12. With our reduction, we did not manage to adjust the gap, materialized by $g(t) - f(t)$ or $g(t)/f(t)$, to be large enough with regard to $f(t)$ and $g(t)$ so that we can derive some non-approximability results.

5. Conclusion

Proving the $\mathcal{NP}$-completeness of the encoding dimension and of the string dimension answers the complexity questions raised by [22] and [24]. This hardness result could be expected due to the closeness of difficult problems like the dimension or the k-dimension. Such a classification encourages to head further researches towards particular classes of orders for which computations could be polynomial or towards heuristics with good average performances or good guaranteed ratios.

It is likely that approximating the encoding dimension with guaranteed ratios depending on $|P|$ is hard. Some non-approximability results have been proved in [27] concerning the k-dimension for fixed k, but for the dimension, this question raised in [40] remains open.

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